ON THE TRANSFORMATION OF THE BOUNDARY IN THE CASE OF CONFORMAL MAPPING.

BY PROFESSOR W. F. OSGOOD.

(Read before the American Mathematical Society, December 30, 1902.)

THE theorem that the interior of any simply connected region may be mapped in a one-to-one manner and conformally on the interior of a circle was stated by Riemann in his doctor's dissertation and was proved by Schwarz and Neumann for the case that the boundary consists of a curve subject to certain restrictions. In the first volume of the *Transactions* * I extended this theorem to the most general simply connected plane region. The generalization is remarkable in that the boundary of such a region need not be a curve, but may be a set of points some of which can not be approached along any curve lying wholly in the region. An extension of the definition of a simply connected region is here obviously necessary. Such a definition may be laid down as follows: By a simply connected region of a plane is meant a continuum such that the whole interior of any closed Jordan curve drawn in the region lies in the region. By a "Jordan curve" is meant a continuous curve without multiple points.

The further question presents itself as to whether boundary points of the region are carried over continuously into points of the circumference of the circle. This question I am able to answer by means of the following theorems. The given simply connected region is denoted by S and it is mapped conformally on the interior of the circle $\mathfrak{S}$.

THEOREM I. *Let A be a point of the boundary of S which can be approached along a curve C lying wholly in S. Then the image of C in $\mathfrak{S}$ is a curve $\mathfrak{C}$ abutting on a single point $\mathfrak{A}$ of the circumference of $\mathfrak{S}$; so that, if the point P approaches A along C, the image $\mathfrak{P}$ of P in $\mathfrak{S}$ will approach $\mathfrak{A}$ along $\mathfrak{C}$.*

THEOREM II. *Let C' be a second curve of S also leading to A, but meeting C only in the initial point of C and in the point*

* "On the existence of the Green's function for the most general simply connected plane region," *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 310.

A. Then the image of C' in $\mathfrak{S}$ will be a curve $\mathfrak{C}'$ abutting on a point $\mathfrak{A}'$ of the circumference of $\mathfrak{S}$. The necessary and sufficient condition that $\mathfrak{A}'$ coincide with $\mathfrak{A}$ is that the closed curve consisting of the two arcs C, C' may be drawn together continuously to the point A without passing out of the region S; in other words, that this closed curve shall contain in its interior only points of S.

Note that *S* consists exclusively of interior points, the boundary not being counted as belonging to the region.

THEOREM III. *If A, A' are two distinct points of the boundary of S which can be approached along a curve C or C' respectively, then the corresponding points $\mathfrak{A}$, $\mathfrak{A}'$ on the circumference of $\mathfrak{S}$ will be distinct.*

THEOREM IV. *Let B be a point of the boundary of S, such that a set of points $Q_1, Q_2 \cdots$ can be chosen in S which have B as their sole limiting point:*

$$\lim_{n=\infty} Q_n = B;$$

but through which no curve can be passed, lying wholly in S and such that a point P describing this curve will approach B as its limit. Suppose furthermore that a curve D can be drawn in S passing through the points Q_n and having but one limiting point A on the boundary of S which can be approached along a curve C, cutting D in every neighborhood of A. Then if the point P describe D, the image $\mathfrak{P}$ of P in $\mathfrak{S}$ will approach the point $\mathfrak{A}$ of the boundary of $\mathfrak{S}$ which corresponds to A.

THEOREM V. *In particular, let S be a region having no boundary points of the kind B considered in Theorem* IV. *Then to each point of the circumference of $\mathfrak{S}$ will correspond a single point of the boundary of S in this sense that if a point $\mathfrak{P}$ approach a point $\mathfrak{A}$ of this circumference along a curve of $\mathfrak{S}$, the image P of $\mathfrak{P}$ in S will approach a point A of the boundary of S, and A will be the same point for all curves of $\mathfrak{S}$ leading to $\mathfrak{A}$. But to distinct points $\mathfrak{A}$, $\mathfrak{A}'$ may correspond the same point A.*

The boundary of S in this case consists of a curve whose coördinates may be represented as single valued continuous functions of a parameter t

$$x = f(t), \qquad y = \phi(t),$$

where t may be chosen as the arc of the circumference of $\mathfrak{S}$, and $f(t)$, $\phi(t)$ are periodic with the length of the circumference of $\mathfrak{S}$ as period.

If furthermore it is impossible to draw a Jordan curve all of whose points but one lie in S and which contains in its interior points of the boundary of S, then the boundary points of S will form a Jordan curve.

The result stated in the latter part of this theorem was recently published by Schoenflies.* In the general case the boundary may be considered as forming a generalization of the Jordan curve; for, although the boundary may pass through a given point more than once, it does not cut itself, and it forms the boundary of a simply connected region.

The method on which the proof of these theorems rests was suggested by physical evidence. The conformal map of S on $\mathfrak{S}$ is effected by the Green's function g of the region S, with pole at O, the existence of g having been established in my former paper (*l. c.*). This function may be interpreted in the well-known manner as the temperature in a certain flow of heat or as the potential in the analogous flow of electricity, the source being at O. The function h conjugate to g can be employed to measure the flux across a given curve; in fact, if c denote an arc drawn in S, the flux across c is proportional to the difference in the values of h at the extremities of c. I was led to the theorems stated above through the following

PHYSICAL LAW.—*Let a fixed circle be drawn in S with O as centre. Let c be an open curve of length l drawn in S, but lying outside the fixed circle. Let the maximum numerical value of the flux across c (or the upper limit, in case no maximum exists), when c assumes all possible forms and positions, be denoted by $f(l)$. Then*

$$\lim_{l=0} f(l) = 0.$$

A rigorous proof of this law may be given by means of the theory of the logarithmic potential function, and proofs of all the theorems here stated will appear later.

HARVARD UNIVERSITY, CAMBRIDGE,
17 *December*, 1902.

* "Ueber einen grundlegenden Satz der Analysis situs," *Göttinger Nachrichten*, 1902, Heft 2.

ON THE QUINTIC SCROLL HAVING THREE DOUBLE CONICS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, October 25, 1902.)

In his paper on quintic scrolls,* Professor Schwarz mentioned that such a scroll exists having three double conics (type IX of those of genus 0). It is now proposed to derive the equation of this surface and to study some of its properties.

A scroll having two double conics will in general be of order 8 and genus 1. In order to generate a quintic scroll the conics must therefore intersect in two points, one of which is a simple self-corresponding point of the (2, 2) correspondence between the points of the two conics, and the other must be a double self-corresponding point. Hence every such quintic scroll must be unicursal. Let the equations of the two conics be

$$c_1: \quad z = 0, \quad x^2 - yw = 0;$$
$$c_2: \quad x = 0, \quad z^2 - yw = 0,$$

respectively. The points of each may be expressed parametrically thus:

$$c_1: \quad x = \lambda, \quad y = \lambda^2, \quad z = 0, \quad w = 1;$$
$$c_2: \quad x = 0, \quad y = \mu^2, \quad z = \mu, \quad w = 1.$$

The equations of a generator which joins the variable point λ to the variable point μ may be written

$$(1) \qquad \mu x + \lambda z - \lambda\mu w = 0, \quad \lambda x - y + \mu z = 0.$$

The line (1) will generate a unicursal quintic scroll having c_1, c_2 for double conics when λ, μ are connected by a relation of the form described. Since the complete double curve of a unicursal quintic is of order 6 it follows that the residual curve is of order 2. Double generators are impossible since the double element in the (2, 2) correspondence is self-corresponding.

* Ueber die geradlinigen Flächen fünften Grades, *Crelle's Journal*, vol. 67.

Two rectilinear directrices can not exist on a scroll having other nodal curves; hence the residual nodal curve is a conic c_3. Since a similar correspondence exists between the new conic and each of the given ones, it follows that the three conics must pass through one common point. No scroll can exist having three double conics which intersect in three distinct pairs of points.

§ 1. *Three Double Conics Having One Common Point.*

Let (0, 0, 0, 1) be the point common to the three conics.

Let c_1 intersect c_3 in $(\alpha, \alpha^2, 0, 1)$, and c_2 intersect c_3 in $(0, \gamma^2, \gamma, 1)$.

The conic c_3 lies in the plane

$$\alpha x - y + \gamma z = 0, \tag{2}$$

and may be expressed as the intersection of (2) and the most general quadric passing through the three given points. The values of $x : y : z : w$ as determined from (1) and (2) are now to be substituted in the equation of the quadric. The result is the desired (2, 2) correspondence between λ, μ. Finally, the result of eliminating λ, μ between this equation and those of a generator is the desired equation of the surface.

The correspondence is of the form

$$\lambda^2 - \mu^2\kappa + m\lambda\mu + \gamma(\kappa + 1)\mu - \alpha(\kappa + 1)\lambda = 0. \tag{3}$$

It has a double element at (∞, ∞) and a simple self-corresponding element at (0, 0), hence the point common to all three conics cannot be a double point. Similarly, the point α is a double point in the correspondence between c_1 and c_3, and γ is a double point in the correspondence between c_2 and c_3. Since the plane of each double conic must contain one generator it follows that three generators pass through the point (0, 0, 0, 1). The equations of these generators are

$$\begin{aligned} x &= 0, \quad \kappa y = (\gamma\kappa + \gamma)z; \\ y &= 0, \quad \alpha x + \gamma z = 0; \\ z &= 0, \quad y = (\alpha\kappa + \alpha)x. \end{aligned}$$

When $\kappa = 1$ they all lie in the plane

$$y = (\alpha\kappa + \alpha)x + (\gamma\kappa + \gamma)z.$$

The residual curve in the plane of the three generators is a conic * c_4; it does not pass through the triple point. This is the only simple conic which can lie on the surface.

Moreover, the scroll is contained in a linear complex and has one algebraic asymptotic line of order 8. If in addition $m = 0$, the complex becomes special and the simple conic breaks up into a simple directrix and a generator. Every plane through the directrix will cut four generators from the surface. The directrix does not intersect any of the double conics, hence the three points in which the three generators from (0, 0, 0, 1) meet the conics again are collinear. The scroll can now be generated by a particular (1, 2) correspondence between the points of the conic and the straight line.

The equations of the directrix are

$$w = 0, \quad y = 2\alpha x + 2\gamma z,$$

and the fourth generator in the plane $y = 2\alpha x + 2\gamma z$ is defined by

$$\gamma x + \alpha z = 2\alpha\gamma.$$

This generator connects the three points in which the generators through (0, 0, 0, 1) cut the double conics again. The special asymptotic line of order 8 is now degraded into the directrix of the complex.

When a unicursal quintic scroll has a double conic the residual nodal curve is in general a twisted quartic with two apparent double points. In general a straight line and a quartic having two apparent double points will determine a scroll of order 8 by the condition that each generator is to cut the line once and the quartic twice. The straight line is a double line on the surface and the quartic curve is a triple line. When the surface is reducible and one factor is a quintic having the line as simple directrix and the quartic for double curve, the other component is a cubic having the straight line and the quartic as simple directrices.

Professor Schwarz remarks that the same thing happens if the quartic be replaced by two conics which have two points of intersection. But such is not the case. Let c_1, c_2 be the conics and l the line. Let c_1, c_2 intersect in $W \equiv (0, 0, 0, 1)$ and in $Y \equiv (0, 1, 0, 0)$. The scroll determined by the common se-

* Mr. C. H. Sisam has recently proved that a simple conic lies on the general scroll. Its plane does not pass through the triple point.

cants of two conics and a line is of order 8, less 1 for each point in which the conics intersect, since the pencils Wl, Yl belong to the system. Our scroll is therefore of order 6 and genus 1, having c_1, c_2, l for double lines. The residual nodal curve is a twisted cubic which is met once by each generator.

If, however, the line l lies in one of the common tangent planes of the two conics the surface will be of order 5, as may be shown as follows: A generator (1) will cut the line

$$A_1x + B_1y + C_1z + D_1w = 0, \quad A_2x + B_2y + C_2z + D_2w = 0$$

if

$$\lambda^2\mu(AB) + \lambda\mu^2(BC) + \cdots + \lambda(CD) + \mu(AD) = 0.$$

This is in general a (2, 2) correspondence of genus 1 with two simple self-corresponding elements at the points of intersection of the conics. If the scroll is to be a quintic, one of these points must be a double element. This condition requires either that the coefficients of $\lambda\mu^2$ and $\lambda^2\mu$ shall vanish, or that the coefficients of λ and μ shall vanish. In the first case the line l lies in the plane $w = 0$, and in the second case l lies in the plane $y = 0$. But these are the common tangent planes of c_1 and c_2 at W, Y. It can easily be verified independently that when $\kappa = 1$ and $m = 0$ the tangent planes of c_1 and c_2 in $(0, 1, 0, 0)$, the tangent plane of c_1 and c_2 in $(0, \alpha^2, \alpha, 1)$, and the tangent plane of c_2 and c_3 in $(\gamma, \gamma^2, 0, 1)$ all pass through the simple directrix.

The only case in which a straight line and two conics which intersect in two points can define a cubic scroll is when the (2, 2) correspondence is factorable, for such a surface is determined by a (1, 1) correspondence between a line and a conic. For example, the conics c_1, c_2 and the line $y = 0$, $x + z = w$ define a reducible (2, 2) correspondence between λ, μ and all lie on the same cubic scroll.

The equation of our quintic, obtained by eliminating λ, μ between (1) and (3), may be written

$$\begin{vmatrix} z^2 - \kappa x^2 - mxz & y + nx & lz - \kappa y & 0 \\ 0 & y + nx & \kappa xy + myz + nz^2 & lz - \kappa y \\ w & 1 & x & 0 \\ 0 & x & z^2 - yw & x \end{vmatrix} = 0,$$

in which $l \equiv \gamma(\kappa + 1)$, $n \equiv -\alpha(\kappa + 1)$.

For the equation of the second surface, having a linear directrix, put $\kappa = 1$, $m = 0$.

§ 2. *Three Double Conics Having Two Common Points.*

Now consider the case in which the third double conic passes through both points of intersection of c_1, c_2. Let

$$x = pz$$

be the equation of its plane. The equation of the general conic lying in this plane and passing through (0, 0, 0, 1), (0, 1, 0, 0) may be written

$$Ax^2 + Bzy + Cyw + Dzw = 0,$$

from which the (2, 2) correspondence becomes

$$B\lambda^2\mu(p+1) + C\lambda^2 p + (A + p^2C + C)\lambda\mu + pC\mu^2 + pD\mu + D\lambda = 0.$$

In order that the correspondence have a double element at one point of intersection of the two conics, and a simple self corresponding element at the other, either $B = 0$ or $D = 0$. This requires that the third conic must touch the common tangent plane of c_1, c_2 at one of their points of intersection. Let $B = 0$. Then the new conic touches the plane $w = 0$ at (0, 1, 0, 0).

The reduced form of the correspondence now becomes

$$\lambda^2 + \mu^2 + m'\lambda\mu + n'\lambda + l'\mu = 0$$

and the analytic procedure is identical with that of the preceding case.

The generators which lie in the planes of the three double conics are respectively

$$x = 0, \quad y + l'z = 0;$$
$$y = 0, \quad x - pz = 0;$$
$$z = 0, \quad y + n'x = 0;$$

since $\kappa = 1$ they cannot lie in one plane. The scroll also contains a simple conic which cuts each of the double conics once. The lines which cut three conics related like c_1, c_2 and the simple conic will in general generate a scroll of order

8. If one scroll be of order 5, the other will be a cubic and not another quintic, as stated by Schwarz.

In this case the surface does not belong to a linear complex; in particular, therefore, *when a quintic scroll has three coaxial double conics it cannot have a rectilinear directrix.*

§ 3. *Two Double Conics Touching Each Other.*

Any two conics which touch each other but which lie in different planes may be expressed by the equations

$$c_1': \quad z = 0, \quad y^2 - xw = 0;$$
$$c_2': \quad x = 0, \quad y^2 - wz = 0.$$

In parametric representation these become

$$c_1': \quad x = 1, \quad y = \mu, \quad z = 0, \quad w = \mu^2;$$
$$c_2': \quad x = 0, \quad y = \lambda, \quad z = 1, \quad w = \lambda^2.$$

The equations of the line joining λ to μ are

$$\mu x + \lambda z - y = 0, \quad \mu^2 x + \lambda^2 z - w = 0.$$

The most general (2, 2) relation between λ, μ which will define a quintic scroll is given by

$$g\lambda^2 + f\mu^2 - \lambda\mu(f + g) + h = 0.$$

The surface has a third double conic c_3' defined by

$$y = 0, \quad hxz + gxw - fzw = 0.$$

The equation of the surface may be written

$$\begin{vmatrix} g & -(f+g)y & hx - fw & 0 \\ 0 & g(x+z) & -(f+g)y & hx - fw \\ z & -2yz & y^2 - wx & 0 \\ 0 & z(x+z) & -2yz & y^2 - wx \end{vmatrix} = 0.$$

The common tangent to c_1', c_2' is a generator. The generator in the plane $y = 0$ is $x + z = 0$. The curve of section made by the plane $x + z = 0$ is

$$zy^2[(f+g)(y^2 + wz) + 2z(hz - fw)] = 0;$$

hence the line $y=0, x+z=0$ is a torsal generator and $x+z=0$ is the torsal plane. The simple conic passes through the point of intersection of the three double conics and touches the generator $x=0, y=0$.

If $f+g=0$ the conic will be factorable. The rectilinear directrix is

$$x+z=0, \quad hz-fw=0.$$

If the simple conic be required to intersect c_3' in a second point, the former must break up.

When the scroll belongs to a linear complex, the complex must be special. A quintic scroll which has three double conics which touch the same line at the same point, does not exist.

Every quintic scroll having three double conics contains either a simple conic or a simple directrix. If in the former case the planes of all four conics pass through the same point, then the plane of the simple conic contains three generators passing through that point.

CORNELL UNIVERSITY,
October 10, 1902.

SURFACES REFERRED TO THEIR LINES OF LENGTH ZERO.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, October 25, 1902.)

CONSIDER a surface S referred to a general system of parametric lines, $u=$ const., $v=$ const. Denote by x, y, z, the cartesian coördinates of a point on S and X, Y, Z, the direction cosines of the normal to the surface. As usual, we put

$$(1) \quad E=\sum\left(\frac{\partial x}{\partial u}\right)^2, \quad F=\sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \quad G=\sum\left(\frac{\partial x}{\partial v}\right)^2,$$

and

$$(2) \quad D=\sum X\frac{\partial^2 x}{\partial u^2}, \quad D'=\sum X\frac{\partial^2 x}{\partial u\partial v}, \quad D''=\sum X\frac{\partial^2 x}{\partial v^2}.$$

When the parameter curves are the lines of length zero,

$$E = G = 0,$$

and the Codazzi equations * reduce to the simple forms

$$\frac{\partial D}{\partial v} - D' \frac{\partial}{\partial u} \log \frac{D'}{F} = 0,$$

$$\frac{\partial D''}{\partial u} - D' \frac{\partial}{\partial v} \log \frac{D'}{F} = 0. \tag{3}$$

The expression for the mean curvature † becomes

$$\tfrac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = -\frac{D'}{F}. \tag{4}$$

From (3) it follows that when S is a surface of constant mean curvature, D is a function of u alone and D'' of v alone. Conversely, when this condition is satisfied, either $D' = 0$ or $D'F$ is constant. In either case the mean curvature is constant; in the former, S is a minimal surface. Hence

THEOREM I. *Surfaces of constant mean curvature are characterized by the property that, when the parametric lines are of length zero, D is a function of u alone and D'' of v alone.*

When any surface is referred to its lines of length zero, the directions of the lines of curvature are given by the equation ‡

$$Ddu^2 - D''dv^2 = 0. \tag{5}$$

In consequence of the above results we have

THEOREM II. *The lines of curvature on a surface of constant mean curvature are given by quadratures when the surface is referred to its lines of curvature.*

By Bonnet's theorem § we know that there is associated with every surface of constant mean curvature, different from zero, a parallel surface of constant total curvature and conversely. Moreover, it can be seen from the general relations ‖ which

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* Bianchi, Lezioni, p. 91.
† *Ibid.*, p. 104.
‡ *Ibid.*, p. 99.
§ Darboux, Leçons, vol. 3., p. 375.
‖ Knoblauch, Einleitung in die allgemeine Theorie der krummen Flächen, p. 234–6.

obtain between the fundamental coefficients of a surface and its parallel that lines of length zero and lines of curvature upon a surface of constant mean curvature correspond to asymptotic lines and lines of curvature respectively upon the parallel surface of constant total curvature. Hence

THEOREM III. *The lines of curvature on a surface of constant total curvature are given by quadratures, when the surface is referred to its asymptotic lines.*

Theorem II is not restricted to surfaces of constant mean curvature, but, as we shall see, is true of all isothermic surfaces. Let S be an isothermic surface; then its linear element can be brought to the form

$$ds^2 = \lambda(du_1^2 + dv_1^2), \tag{6}$$

where λ is a function of u_1 and v_1, and the curves $u_1 = \text{const.}$, $v_1 = \text{const.}$ are the lines of curvature. Now the lines of length zero are given by

$$du_1^2 + dv_1^2 = 0,$$

that is,

$$u_1 + iv_1 = 2u, \quad u_1 - iv_1 = 2v, \tag{7}$$

where u and v are the parameters of the lines of length zero. From (7) it follows that, if the surface is referred to lines of length zero, the equation of the lines of curvature is

$$du^2 - dv^2 = 0,$$

and for a general choice of parameters this would be

$$Udu^2 - Vdv^2 = 0,$$

where U is a function of u alone and V is a function of v alone. Comparing this equation with equation (5) we get

$$\frac{D}{D'} = \frac{U}{V}. \tag{8}$$

Hence

THEOREM IV. *When an isothermic surface is referred to its lines of length zero, the lines of curvature are given by quadratures.*

And conversely

THEOREM V. *When a surface is referred to its lines of length zero and the condition* (8) *is satisfied, the surface is isothermic.*

In consequence of the preceding results the equation whose

solution gives all surfaces of constant mean curvature can be reduced to a simple form. By a suitable choice of parameters we have $D = D' = 1$, and there is no loss in assuming that the square of the mean curvature is unity. When these values are substituted in the Gauss equation,* it reduces to

$$\frac{1}{F}\frac{\partial^2 \log F}{\partial u \partial v} = \frac{1}{F^2} - 1.$$

If we put $\log F = \theta$, this equation becomes

$$\frac{\partial^2 \theta}{\partial u \partial v} = e^{-\theta} - e^{\theta}.$$

When a solution of this equation is known, all of the fundamental coefficients of the surface are given.

PRINCETON, N. J.,
September 20, 1902.

SUPPLEMENTARY NOTE ON THE CALCULUS OF VARIATIONS.

BY DR. E. R. HEDRICK.

IN a recent paper in the BULLETIN,† the present writer stated the theorem: "The condition $\partial^2 f/\partial y'^2 > 0$ for all x, y on the curve C, and for all y' considered, is a sufficient condition for a minimum" (of the integral $\int f(x, y, y')dx$ along the curve C), provided that certain preliminary requirements are satisfied. This theorem stands in apparent contradiction with an example given by Professor Bolza;‡ and while the contradiction is *only* apparent, it seems fitting to point out clearly the actual agreement of my results with those of Professor Bolza.

In the article mentioned I have required (page 11) that the integral $\int f(x, y, y')dx$, shall be considered only in a region R (of the x, y, y' space), such that the integrand $f(x, y, y')$ is

* Bianchi, *l. c.*, p. 67.

† "On the sufficient conditions in the calculus of variations," BULLETIN, vol. 9 (2), no. 1 (Oct., 1902), p. 15.

‡ "Some instructive examples in the calculus of variations," BULLETIN, vol. 9 (2), no. 1 (Oct., 1902), Example II, p. 9.

analytic at each point of R, *i. e.*, for all sets of values of x, y, y' in R. A region is here a closed manifold of points, *i. e.*, it possesses a boundary, so that a point upon the boundary of R belongs to the region. Hence for the reasoning throughout it is necessary to choose such a region R. It is true that this restriction is by no means necessary for most of the results of the paper, but without it some of the proofs, and in particular the above theorem, would have to be modified. It is to be noticed, however, that we may choose an *unbounded* region R_1 (*i. e.*, exclusive of the boundary) provided R_1 lies wholly within a region R such as is required above, and that all the results of the paper hold when the given integral is considered only for points *inside of* R_1, or in particular *inside of* R itself. In other words, the results given hold in any region (inclusive or exclusive of the boundary) provided the points for which the integrand $f(x, y, y')$ is not analytic be excluded from the region (*i. e.*, from its interior and from its boundary).

In Professor Bolza's example, I would then, for the purposes of my paper, *first* choose as my region R the region

$$R: \qquad |x| \leqq A; \quad |y| \leqq A; \quad |y'| \leqq A.$$

Any positive number A having been chosen for the example once for all, the investigation of minima depends on our ability to find, *secondly*, a value for δ so small that the supposed solution renders the integral value less than along any other curve for which

$$|\eta(x)| < \delta, \qquad x_0 \leqq x \leqq x_1.$$

It is seen at once that any *order of choice* of these quantities A and δ corresponds exactly to the same order of choice of the h and k of Professor Bolza's article, respectively; whereas the order of choice necessary for the considerations of my paper is exactly opposite to that made by Professor Bolza. For, from my standpoint, k cannot be *first* chosen and then kept fixed while h (*secondly*) approaches zero, as is necessary for Professor Bolza's argument. A choice of h *after* k has been chosen and fixed, necessitates a choice of $A \geqq k/h$. This I would regard as the first step, after which I would have δ at my disposal, which would necessitate a re-choice of k, if δ be chosen less than k. It readily follows that, from my standpoint, the conclusion $\Delta I < 0$ cannot be drawn, and that the X axis actually renders

the integral in question a strong minimum,* in the above region R, *i. e.*, in any region of values of x, y, y' from the interior and boundary of which the values $x = \infty, y = \infty, y' = \infty$ are excluded. Accordingly the example cited, as also Professor Bolza's theorem at the top of page 9, are perfectly reconcilable with my theorem quoted above.

The integrand in question is discontinuous for infinite values of either x or y or y', and in fact becomes infinite to the second order for $y' = \infty$. It may be pointed out that the same apparent contradiction will arise in the case of any integral whose integrand becomes infinite to the second order for any value of y' (not necessarily $y' = \infty$); but again the contradiction is only apparent. It might be questioned whether simple ordinary integrals, such as the integral of length, $\int \sqrt{1 + y'^2}\, dx$, do not present similar phenomena, so that the results of my paper would be restricted in their application, in all ordinary examples, to a closed region (of the x, y, y' space) in which the integrand is analytic. Such are indeed the formal requirements of the paper. But it may be noted that if the transformed integral

$$\int f\left(x, y, \frac{1}{x'}\right) x'\, dy,$$

where $x' \equiv dx/dy$, be analytic for the directions previously excluded ($y' = a, x' = 1/a$), then the whole reasoning applies to the comparison curves previously excluded, by a simple process of transformation and subdivision of the curve, at least provided that the comparison curves in question satisfy the simple requirement of not being parallel to each axis an infinite number of times in any interval.

And further, the requirement made on page 11 is by no means necessary for the proof of most of the theorems. In view, however, of the aims of the article, it was deemed advisable to include this restriction, in order to simplify the proofs, and in order to arrive at the theorem mentioned above without too extended a discussion of possible alternate restrictions.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
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*The distinction between strong and weak minima holds as usual because the slope restriction $|y'| \leqq A$ was made before any particular extremal was found, and does not regard any particular extremal.

THE SYNTHETIC TREATMENT OF CONICS AT THE PRESENT TIME.

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(Read before the American Mathematical Society, December 29, 1902.)

In the August-September number of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* for the year 1902 Professor Reye has printed an address, delivered on entering upon the rectorship of the university at Strassburg, on "Synthetic geometry in ancient and modern times," in which he sketches in a general manner, as the nature of his audience necessitated, the differences between the methods and points of view in vogue in geometry some two thousand years ago and at the present day. The range of the present essay is by no means so extended. It covers merely the different methods which are available at the present time for the development of the elements of synthetic geometry — that is the theory of the conic sections and its relation to poles and polars.

There is one method of treatment which was much in use, the world over, half a century ago. At the present time this method is perhaps best represented by Cremona's Projective Geometry, translated into English by Leudesdorf and published by the Oxford University Press, or by Russell's Pure Geometry from the same press. This treatment depends essentially upon the conception and numerical measure of the double or cross ratio. It starts from the knowledge the student already possesses of Euclid's Elements. This was the method of Cremona, of Steiner, and of Chasles, as set forth in their books. That was fifty years ago.

In the year 1866 Professor Reye, mixing with a bold and somewhat original hand the newest ideas of von Staudt with those of Steiner and Poncelet, older and better known, published his Geometrie der Lage which owing to its incomparably clear and pedagogic style has been the mainstay of synthetic projective geometry these many years, slowly displacing the method of Cremona, and somewhat obscuring the elegance of von Staudt's.

At present, to judge from the text-books upon the market and the announcements of courses in the universities, the Ital-

ians have not only forsaken Cremona but are persistently edging by Reye toward the long neglected von Staudt; the Germans, pulled with almost equal forces backward by Steiner and forward by von Staudt, still cling to the intermediate, Reye. In France, where no one studies that which there is not some one great genius to teach, synthetic geometry in its higher aspects has seemingly perished by the way with Chasles—for what was it to Hermite and Bertrand or is it now to Darboux and Picard? In England and America Cremona is still much followed, and when not Cremona, then Reye.

In view of these facts, no more will be said of the metric, cross ratio method of developing synthetic geometry. That method caught its fatal chill when von Staudt published his book in 1847. It has been shaking ever since. Some day it will die. Not long hence even the fondest of us must recognize it as quite a thing of the past.

The newer methods are all founded ultimately upon the work of von Staudt, the essential part of which appeared in 1847 as has been mentioned, although the addition of three supplements stretched across the following thirteen years. These methods are more powerful and beautiful because they deal more directly with the ideas necessarily involved in synthetic projective geometry. They are more natural and satisfactory from a purely geometric standpoint. They are more compact and at the same time they are more far-reaching. In point of fact they are all nothing but variations on von Staudt's original, according as it is more or less fully adhered to, according as a less or a greater amount of the earlier methods are interwoven with it.

The chief variations, those which we shall discuss here, are three. We shall designate them as follows:

1° The extreme, which may be seen expounded in Reye's "Geometrie der Lage," 4th edition, volume I, Leipzig, or in Holgate's translation, New York.

2° The intermediate of which an excellent and elementary exposition is given in Böger's "Ebene Geometrie der Lage," Leipzig, or combined with the geometry of space in Sannia's "Geometria proiettiva," Naples.

3° The slight variation, or none at all, such as is presented in Enriques's most excellent "Geometria proiettiva," Bologna, or to a much less extent in a paper in the *Annals of Mathematics*, second series, volume 2, by Miss C. A. Scott.

By a glance at the books mentioned, these three methods may be seen to have in common a certain considerable number of elementary ideas and theorems; namely, the law of duality, the theorem of Desargues, the construction of harmonic elements, the theory of the projective relations between the points in two ranges or in the same range counted twice. These are explained perhaps most simply and succinctly by Böger. To them may be added, if desired, the elements of the theory of collineations and correlations between the points and lines of two planes or of the same plane counted twice.

From this point on, the three methods diverge. In the first a conic is defined as the locus of the intersection of corresponding rays in two projective, but not perspective, pencils. The usual theorems concerning inscribed and circumscribed quadrilaterals, Pascal's and Brianchon's theorems, and so forth, are then proved. The polar of a point with respect to a conic is defined by means of the well known harmonic property. The theory of poles and polars follows, completing the usual elements of conic sections from the synthetic standpoint.

The advantages of this method, the reasons why it holds its popularity so easily, are not hard to see. The processes seem direct. The construction of a conic is a pretty problem, a pretty exercise in draughting, for the student. (It is however by no means so pretty a problem to prove that a conic is independent of the two particular points upon it which may have been chosen as the vertices of the defining projective pencils.) A ruler and pencil are real, tangible things which are good for the student to use constantly. Moreover the idea of polar, introduced by means of the harmonic property, is easily grasped. The thing itself is easily constructed.

But there are disadvantages, too, of which one is that if the pole be outside the conic the construction for the polar yields only a segment of a line, only the part of the polar inside the conic. This is inconvenient. It introduces a sort of lack of symmetry which sticks out awkwardly in some proofs unless, drawing on the analogy furnished from analytic geometry, we make a bold application of the so-called principle of continuity. Again there is the awkward lack of symmetry mentioned in the parenthesis above. There are other disadvantages which will appear when the advantages of the third method are pointed out.

In the second method the conics are defined asbefore. To define the relations of pole and polar, the conception of an

involution upon a conic is used. Points $A, B, \cdots; A', B', \cdots$ upon a conic are said to be in involution if the relation between them is projective in such a manner that if A corresponds to A', then A' corresponds mutually to A. It is shown that the lines which join corresponding points of an involution meet in a point P and that tangents drawn at corresponding points meet in a line p. The point P and line p are defined as pole and polar.

This method is not much better than the first. It is but a timorous intermediate between it and the third. It has nothing to recommend it but the drilling of the student in the ideas of projectivities and in the special case of involutions. These statements must not be construed as in any way detrimental to Böger's book, which is, on the contrary, far from tiresome and contains in its later portions a complete presentation of the newest ideas in adjoined and resultant involutions and in polar fields. In fact the standpoint, except at the commencement, is that of the third method of treatment.

The third method sets about matters in a wholly different way. Proceeding from the common ground before mentioned it lays great stress on the idea of projective relations between ranges or pencils and of collinear and correlative relations between two planes whether different or coincident. Here as in the first method the student may have plenty of use for his ruler and pencil. The subject only is different. Here he takes four points A, B, C, D and four other points A', B', C', D' and constructs corresponding points P and P' in the collineations thus defined, or he takes four lines a, b, c, d and constructs correlative points and lines P and p or inversely q and Q. These exercises are but preliminary. The important case is that of a *correlation which is involutory*, in which if P is correlative to p, then inversely p is correlative to P. Such an involutory correlation is called a polarity. The correlative points and lines are the poles and polars of the polarity.

As yet conics have not been defined. The definition is given as follows: The locus of those points in the polarity which lie upon their own polars (*if such points exist*) is a conic. Or the envelope of those lines in the polarity which pass through their own poles (if such line exist) is a conic. These two definitions lead to the same curve. The demonstrations that this curve is a conic according to the other definition of methods 1° and 2°, that the poles and polars with regard to the polarity are poles

and polars with regard to the conic, and all the usual theorems concerning conics and poles and polars follow immediately with great directness and simplicity.

The disadvantage which has been attributed especially to this method is its difficulty. There certainly is difficulty in the method either for the student or for the instructor. If the latter is content to follow hard in the footsteps of von Staudt or even directly after Enriques or C. A. Scott the student will have difficulty. This method has not been so trampled smooth, so laid out with problems as the older. But if the instructor will only apply himself to obtaining graded exercises for the student, if he will be content to go slowly and carefully, he will find that in the end he will have covered as much ground as easily and comprehensibly to the student, as if he had kept to either of the other methods.

And the advantages, whether on the score of beauty and power or on that of the insight afforded to the student into the subject of projective geometry, are alike great. For it must have been observed that the fundamental difference in the three methods is the relative importance in them of the idea of the transformation—the projective transformation, whether collineation or correlation—and in particular the involutory projective transformation. In the first method only so much knowledge of transformations is needed as will enable one to associate corresponding rays in two pencils, in the second the elementary properties of involutions on a conic are required, in the third the important fundamental notions of collineation and correlation are indispensable.

For the last thirty years, or to be more precise, since the publication of Klein's Erlangen Programm in 1872, there has been no reason why geometers should not recognize distinctly —although they have often been slow to do so— that the fundamental thing, next to the axiom, is the transformation or groups of transformations. In projective plane geometry, whether the treatment be analytic or synthetic, the important thing is the group of collineations G_8 plus the correlations Γ_8. The conic, a projective concept in itself, belongs essentially to this group and may be treated most appropriately in connection with it. Von Staudt's great achievement was to see this and publish it in 1847, a quarter century before the appearance of Klein's Erlangen Programm. It is for the geometers who follow him so to arrange his method that it shall be easily acces-

sible to students without being changed so essentially as to lose its chief vantage points.

We may note in passing that in other fields than projective geometry as much if not greater sloth in adopting the transformation theory and its benefits is exhibited. How many a weary page of Euclid's elements could be brightened by the introduction of the theory of displacements in connection with polygons and polyhedra and of the theory of inversion in connection with circles and spheres. How many a useless proposition could be replaced by one full of promise for future applications.

To return to projective geometry. One advantage of von Staudt's method has been seen to be that it places in evidence the importance of the projective group. There is another advantage which might at first seem a disadvantage. It is this. The conic has been defined subject to its existence. There is a large class of polarities which yield no conic. Analytically they would yield an imaginary ellipse of the type

$$a^2x^2 + b^2y^2 + c^2 = 0.$$

But in these polarities a great number of theorems, real in every detail, hold just as in polarities which yield a real conic. For example, the theorem that the six vertices of two triangles self-conjugate in the same polarity lie on a conic is evidently real and valid for any sort of polarity. So with other theorems.

It is evident, therefore, that the third method, which in all essentials is von Staudt's, affords a treatment of polarities and consequently of conic sections which is independent of the reality or non-reality of those conic sections, provided only that the polar field defined by them is real. This means that we have a synthetic method for treating any conic section which would be representable analytically by an equation with real coefficients. Not often is a greater generality assumed for analytic methods.

The salient advantages of von Staudt's method are then these: First, insistence on treating projective geometry from the standpoint of the projective group. Second, ability to treat in a similar manner all conics, real or imaginary, which define real polar fields. The disadvantages are a supposed difficulty and indirectness. The latter does not exist and the former would reduce almost to nothing if half the attention were expended on the best method which has been spent on the worse.

I say the best method, referring to von Staudt's. It certainly seems at present the best not only of those known, but of any which may be invented. For what more or better could be expected of any purely synthetic method than that which is afforded by the two above mentioned salient advantages of this. It seems scarcely possible that any synthetic method should treat a more general class of conics or treat them in a more concise and germane manner.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
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BROWN'S LUNAR THEORY.

An Introductory Treatise on the Lunar Theory. By ERNEST W. BROWN, M.A. Cambridge, Eng., The University Press, 1896. xvi + 292 pp.

THE lunar theory, besides containing inherent difficulties of a very serious character, involves such a mass of details that it is only with a great deal of labor that one can get a satisfactory idea of it from the original memoirs. The essential relations and differences of the various methods are obscured as they come from the hands of their authors by the differences in the choice of variables and notations. In view of the intrinsic value of the subject and of its great importance as the best test of the newtonian law, and the fact that advances in it can hardly be hoped for from one who is not familiar with what has been done in the past, the desirability of a treatise starting at the very foundations, pointing out the difficulties which are encountered and the methods which have been used to overcome them, and giving the essentials of the most important processes employed by the various investigators without including the almost endless details, can easily be appreciated. This book has evidently been written to fill this need, and it may be said at once that Professor Brown has attained a very high degree of success.

The only other book of at all the same character is the third volume of Tisserand's Mécanique céleste, which is devoted to the theory of the motion of the moon. Tisserand's ideal is somewhat different, in that he aims to give a fairly complete account of all the lunar theories of particular merit which have

been developed. Moreover, his third volume is not complete in itself, as the results of the first two volumes are used so far as they are needed.

An idea of the contents and arrangement of Professor Brown's treatise can be obtained from the titles of the chapters, which are in their order: Force functions, The equations of motion, Undisturbed elliptic motion, Form of solution, The first approximation, Variation of arbitrary constants, The disturbing function, De Pontécoulant's method, The constants and their interpretation, The theory of Delaunay, The method of Hansen, Method with rectangular coördinates, The principal methods, Planetary and other disturbing influences. In the table of contents a complete enumeration of the topics treated is given, and this is supplemented by a general index, an index of authors, a reference table of notation, a general scheme of notation, and a comparative table of notation. From this it is seen that the work has been arranged so as to make its reading as simple as possible for the beginner, and to make it a valuable reference work for those who may be familiar with the subject.

The care with which the preliminary ground has been gone over has put the actual introduction of the perturbations due to the sun far along in the book. Thus, the reader finds himself compelled to read six chapters before he gets to that which he is continually expecting. It might, perhaps, have been better to have put the long chapter on the variation of constants after the chapters on de Pontécoulant's method and the constants and their interpretation. This would have remedied to a considerable extent that which in any case is apt to strain the patience of the reader. Aside from the possible modification just mentioned, the general plan of the book could scarcely be improved.

There are three essentially different types of lunar theory — that of de Pontécoulant, that of Delaunay, and that first developed by Hill, to which may perhaps be added that of Hansen as containing many features of more or less importance different from the others. That of de Pontécoulant and most of his predecessors consists in developing certain coördinates in periodic series of assumed form with the time or true anomaly as argument and determining the coefficients step by step as powers of the small parameters involved; that of Delaunay consists in applying the method of the variation of parameters in the

canonical form over and over in such a way as to remove the most important parts of the perturbative function; that of Hill consists in finding very accurate particular solutions of the differential equations after the parts depending on the parallax of the sun, the eccentricity of the earth's orbit, and the latitude of the moon have been neglected, and then finding the deviations from this orbit due to general initial conditions and the neglected part of the perturbative function. The author has wisely treated these three types and the method of Hansen in detail, and explained the others briefly by their relations to them.

A number of typographical errors have crept in, but this could scarcely be avoided in a book involving so many complicated formulas. Those which have been noticed will cause the reader no trouble. There are some other points to be mentioned, having their origin for the most part in the imperfect state of the lunar theory from a mathematical standpoint rather than in faults of presentation of well-established methods. Thus, in the exposition of de Pontécoulant's method, which is somewhat different from the original yet fairly representative of methods of this type, the treatment of the constants of integration leaves something to be desired. The differential equations for motion in the plane of the ecliptic are, if $r = 1/u$ is the radius of the moon's orbit and v her longitude,

$$\frac{\frac{1}{2}d^2\left(\frac{1}{u^2}\right)}{dt^2} - u + \frac{1}{a} = m^2 P_1(u, v, e', t),$$

$$\frac{dv}{dt} - hu^2 = m^2 P_2(u, v, e', t).$$

a and h are constants depending on the initial conditions, m is the ratio of the mean motion of the sun to that of the moon, and e' is the eccentricity of the earth's orbit. If the right members are neglected, ordinary elliptic motion is obtained. In this, u is a periodic series whose coefficients are polynomials in the eccentricity of the moon's orbit, their structure depending uniquely on the form of the differential equations.

If the coördinates have the form $u = u_0 + \delta u$, $v = v_0 + \delta v$, where δu and δv are the perturbations, it is found after substituting in the differential equations that the main part of δu is defined by the equation

$$\frac{1}{n^2}\frac{d^2\delta u}{dt^2} + \delta u = m^2 Q(t).$$

The general solution of this linear equation consists of two parts, the particular integral and the complementary function. The period of the complementary function is the same as that of the first periodic term in the elliptic motion. For this reason (see pages 50 and 117) the complementary function is omitted, or rather is supposed to be included in the elliptic term. This is of course allowable since the coefficient of the first elliptic term is arbitrary, but it is not evident that the coefficients of the higher multiples of the elliptic argument are related to this *modified* coefficient of the first term precisely as they are in undisturbed motion. This seems to be assumed tacitly, and it is to be regretted that the reason for it is not given if it is true. The fact is that there are but six arbitrary constants in the solutions of a system of differential equations of the sixth order, and it should be clearly shown how all of the constants can be expressed in terms of the six which are taken as the arbitraries. Some of the statements in Chapter VII might lead a careless reader to infer that many constants may be taken arbitrarily. It would have been clearer if the statement had been made that the constants $E + B + \cdots$ and similar combinations should in the final results be set equal to a constant e_1 rather than that any of them should arbitrarily be set equal to zero. Such a method would express all of the constants in terms of six arbitraries; at the end transformations would be made so as to give six coefficients any desired form. This is the method clearly carried out by Delaunay.

Chapter VIII, which is devoted to a discussion of the meaning of the constants (that is, the coefficients of the principal terms) and the manner in which they are determined, is much to be commended. The treatment of the methods of Delaunay and Hansen is also quite satisfactory.

Chapter XI treats of the methods inaugurated by Hill in his celebrated Researches, and of the extensions to the terms of higher order which the author has himself carried out in numerous important memoirs. It is unfortunately called the "Method with rectangular coördinates." In the first place all of the earlier methods are named after their respective authors and there is no apparent reason why it should not be done here, especially in view of the fact that its whole spirit is

radically different from that of any previous method. In the second place the title adopted in no fundamental way characterizes the method. It is true that the differential equations are at first written in rectangular coördinates where the axes rotate uniformly; but before the integration is begun a transformation is made to complex variables. It is clear that the rectangular coördinates are only incidental and that the final equations in the complex variables might be derived without using them at all. Hill's first problem was to find very exact particular solutions of the chief parts of the differential equations, and in doing this he used two important artifices. The first was to remove the explicit presence of the time in the differential equations by rotating the axes; the second, to reduce the differential equations to the bilinear form by the use of complex variables in order that the task of finding the coefficients of the solutions might not be hopelessly involved. There is no trouble in finding the coefficients when the solutions of differential equations are developed as power series in the independent variable, or in parameters, but the general problem of finding the coefficients when the solutions are Fourier series presents great difficulties. Hill overcame them with rare ingenuity in this problem. If it is desired to give the method a title which characterizes it the method of the variational orbit might be used, but the reviewer suggests as being more appropriate that it be called the method of Hill, or the Hill-Brown method.

In developing the particular solutions of the restricted differential equations Brown, and Tisserand also, infer their form from the results of the earlier methods in the lunar theory. These methods, besides not having been proved to give convergent series, lead to expressions of the same form for general initial conditions. Now the particular solutions found depend upon particular initial conditions for their existence. Hill correctly inferred their character, if they exist, from the form of the differential equations; and he proved their existence by the actual construction of certain nearly periodic orbits by mechanical quadratures and an appeal to the analytic continuity of curves when considered as functions of the initial values of the variables. It seems clear that Brown and Tisserand have taken a backward step in this matter, for it is evident that assumptions should be introduced as sparingly as possible, and that an ideal method is complete in itself.

The discussion of the terms which depend upon the ratio of the periods of the moon and sun and the first power of the eccentricity of the moon's orbit is in excellent shape for one to get from it a practical command of the ideas which Hill developed in his memoir in the *Acta Mathematica*. Yet from a theoretical point of view it would have been better to have proved the existence of the periodic solutions and their form, or at least to have referred to places where such proofs are given, rather than to have inferred their existence and character from earlier theories. The equations which give these terms are linear and non-homogeneous with periodic coefficients and the properties of their solutions are now well known. It might also have been remarked that there are other methods of finding the coefficients and the constant which determines the part of the motion of the moon's perigee which is of the first order in the eccentricity.

A very interesting phenomenon occurs in the literal expansions of the coefficients of the variational terms. If the expansions are made in powers of m (m equals about $\frac{1}{13}$), where m is the ratio of the mean motion of the sun to that of the moon, the series converge very slowly. But if the expansions are made in powers of m' where $m' = m/(1 - m/3)$ (therefore $m' = \frac{3}{38}$ nearly) the series converge with great rapidity. It is at first astonishing that so slight a change in the parameter should have such important results.

The most important parts of these coefficients come from the expansions of $(6 - 4m + m^2)^{-1}$. Hill inquires what value of α must be substituted in the equation $m' = m/(1 + \alpha m)$ so that the series in m' shall converge most rapidly. Hill says, and Brown repeats the statement, that it is easily found that α must be given the value $-\frac{1}{3}$. It is not stated just what is meant by rapidity of convergence, but if the remainder after a given number of terms is to be a minimum the statement is not perfectly correct, although $-\frac{1}{3}$ is undoubtedly the most convenient value to give α in the case in hand. If m had a value large enough the series would actually converge more slowly after the transformation.

The radius of convergence of the series in m is $\sqrt{6}$, and in m', when $\alpha = -\frac{1}{3}$, $\sqrt{18}$. If m has such a value that $m' > \sqrt{3}\, m$ the new series will converge less rapidly than the old. The value of α which must be used to secure a minimum for the remainder depends upon both the numerical value of m and

the number of terms taken. Its value can be determined nearly from the formulas for the remainder limited by inequalities.

Let $f(m) = (6 - 4m + m^2)^{-1}$ and $\phi(m')$ be the same function after the substitution $m' = m/(1 + \alpha m)$. Let ρ represent the true radius of convergence of $f(m)$ when expanded as a power series in m, and ρ' the corresponding radius when $\phi(m')$ is expanded as a power series in m'. Let ρ_1 be a real positive number such that $\rho_1 < \rho$. Let M be the maximum value of the function $f(m)$ on the circumference of the circle whose radius is ρ_1. As m takes all the values on the circumference of this circle m' will take all the values on the circumference of a circle whose radius may be represented by ρ_1'. The maximum value of $\phi(m')$ on this circumference is also M. Let $R_n(m')$ be the remainder of the series in m' after the first n terms have been taken. Then $R_n(m')$ is limited by the inequality

$$R_n(m') < \frac{M\left(\frac{m'}{\rho'}\right)^n}{1 - \frac{m'}{\rho'}} \text{ for all } |m'| < \rho_1'.$$

Consider now the problem of making $R_n(m')$ a minimum. Let ρ_1 be taken as the numerical value of m; hence M is defined by the given value of m and the form of the function $f(m)$, and is a constant with respect to the proposed transformation. Without an absolute minimum being insured because of the presence of the inequality, $R_n(m')$ will generally be smaller the smaller the fraction $(m'/\rho')^n/(1 - m'/\rho')$ is made. This fraction depends upon α as the arbitrary. Let it be represented by $F(\alpha)$. The necessary condition for a minimum is $\partial F(\alpha)/\partial\alpha = 0$. In order to compute this derivative, m' and ρ' must be expressed in terms of α. From the equation of transformation $m' = m/(1 + \alpha m)$ and therefore $m = m'/(1 - \alpha m')$ and

$$(6 - 4m + m^2)^{-1} = (1 - \alpha m')^2[6 - 4(1 + 3\alpha)m' + (1 + 4\alpha + 6\alpha^2)m'^2]^{-1}.$$

The only singularities are poles and it follows at once that

$$\rho' = \frac{\sqrt{6}}{\sqrt{1 + 4\alpha + 6\alpha^2}}.$$

Now forming the partial derivative, casting out the common factors, and clearing of fractions it is found that

$$12n(1+3\alpha)(1+\alpha m)^2 - 6nm(1+\alpha m)(1+4\alpha+6\alpha^2) - 2\sqrt{6}m(n-1)(1+3\alpha)(1+\alpha m)\sqrt{1+4\alpha+6\alpha^2} + \sqrt{6}(n-1)m(1+4\alpha+6\alpha^2)^{\frac{1}{2}} = 0.$$

When $m=0$ then $\alpha+\frac{1}{3}=0$ is a simple solution of this equation. Therefore α may be represented as a power series of the form

$$\alpha = -\tfrac{1}{3} + a_1 m + a_2 m^2 + \cdots$$

which converges for sufficiently small values of m. Substituting this expression in the preceding equation and equating to zero the coefficients of the various powers of m it is found that

$$a_1 = \tfrac{1}{18} - \frac{\sqrt{2}(n-1)}{108},$$

.....

whence

$$\alpha = -\tfrac{1}{3} + \frac{m}{18} - \frac{\sqrt{2}(n-1)\,m}{108} + \cdots.$$

Consequently $-\frac{1}{3}$ is the limit of the value of α as m approaches zero. As m is very small in the lunar theory this value of α is very satisfactory, and particularly so as it changes the trinomial into a binomial.

To illustrate the matter consider a numerical example with a large value of m, say $m=\frac{1}{2}$. Then let the remainder be considered after three terms are taken, or $n-1=3$. It follows that $\alpha=-.318650$, $m'=.594760$, and $(6-4m+m^2)=.235294$. Now it is easily found that

$$[6-4m+m^2]^{-1} = \tfrac{1}{6}(1+\tfrac{2}{3}m+\tfrac{5}{18}m^2)+R_3(m),$$

and when $\alpha=-\frac{1}{3}$

$$[6-4m+m^2]^{-1} = \tfrac{1}{6}(1+\tfrac{2}{3}m'+\tfrac{1}{18}m'^2)+R_3'(m'),$$

and when $\alpha=-.318650$

$$[6-4m+m^2]^{-1} = \tfrac{1}{6}(1+\tfrac{2}{3}m'+.065345m'^2)+R_3(m').$$

It is found by substituting the numerical values that

$$R_3(m) = \quad .001498,$$
$$R_3'(m') = -.001373,$$
$$R_3(m') = -.001310.$$

These few terms are only enough to illustrate the matter.

The object of a literal expansion of the coefficients is doubtless to enable one to compute them most conveniently for various values of the parameter which may arise in practice. In the lunar theory, where alone there is any demand for extended computations, the parameter m is known with a high degree of accuracy. If it were exactly known there would be no occasion for these expansions for use in the lunar theory, since the power series possess no theoretical advantages over the series of fractions either when retained literally or when reduced to numbers.

If the numerical value is given to m on the start the labor of computing the coefficients is enormously reduced, but they admit of no direct corrections for improved values of the parameter. But if m_0 represents the approximate value of m which is at present known, and if the substitution $m = m_0 + m'$ be made, a series will be obtained in m' which will converge extremely rapidly because of the smallness of this parameter. A very few terms will be sufficient to give the results correctly to many decimals for any value of m' which is apt to appear necessary to use. For practical application in the lunar theory this preserves all the generality attained in using m and secures most of the convenience of using numbers from the start. The unfavorable features are that the numerical coefficients cannot be expressed as ordinary fractions and the trinomial $(6 - 4m + m^2)$ remains a trinomial after the transformation. However, if the substitution

$$m = \frac{m_0 + m'}{1 + \alpha m'}$$

is used, the α may be determined so that the trinomial shall reduce to a binomial after which the expansion may be made with great ease. For values of m far different from m_0 the series in m' converge about as rapidly as those in m.

Chapters XII and XIII, which treat briefly of the principal theories other than those considered earlier in the text and of

the effects of planetary and other disturbing influences, are much to be commended except the paragraph in the center of the last page. In this the astonishing statement is made that, because the moon's orbit shrinks as the eccentricity of the earth's orbit decreases, the moon would fall on the earth if the earth's orbit became circular. Obviously a variable may always decrease without having zero as its limit, and in this case the incorrectness of the conclusion is quite evident.

In recapitulation it may be said that Professor Brown has written a very satisfactory treatise on the lunar theory, the faults being for the most part those which have their origin in the inherent, and up to the present unsolved, difficulties of the subject. There is no better source of information for one who wishes to acquaint himself with this celebrated field of investigation.

F. R. MOULTON.

THE UNIVERSITY OF CHICAGO,
November, 1902.

THE DOCTRINE OF INFINITY.

Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. Von Dr. K. GEISSLER. Leipzig, Teubner, 1902. 4to, 417 pp.

THE recent tendency toward critical investigation of the axioms underlying mathematical sciences has led to the necessity of discussing the philosophical basis of the whole mathematical structure, and it is to be hoped that more attention will be given the subject from a purely philosophical standpoint.

The announcement of a book treating the old stumbling block of the infinite from both the mathematical and the philosophical points of view was therefore of the greatest interest, and the character of the book was apparently guaranteed by the excellent reputation of the publishers. But the book — though not lacking in a certain kind of interest — is in several respects most surprising, and, as we shall see, somewhat disappointing.

The first 296 pages are devoted to mathematical investigations, and we shall consider these alone. A great variety of topics are treated, such as the parallel axiom, indeterminate forms, tangents, limits, derivates, velocity, curvature and many

other problems involving continuity. It is not here proposed to discuss these subjects in detail, but to confine ourselves to one serious matter, which the author regards as fundamental and which certainly affects the whole of his reasoning. This is his introduction of certain (fixed) infinitely great and infinitely small numbers (or segments in geometry).

There are two reasons for considering the subject seriously. In the first place it is common, in certain elementary treatises on the calculus, to introduce such infinitely small quantities and to deal with them under the name of "differentials." From the standpoint of mathematical pedagogy, it is certainly of vital importance to discover to what conclusions the introduction of such numbers leads, both with regard to the correctness of using them at all and with regard to the effect upon the student's conceptions and his mode of thought. Secondly, from the point of view of the foundations of mathematics, it is interesting to investigate the axioms tacitly assumed and to see just what axioms (of the usual set) may be retained and what new axioms need be added in the logical treatment of such infinitely great and infinitely small quantities, capable of assuming fixed constant values.

One of the axioms generally accepted in both arithmetic and geometry is that of Archimedes, which may be stated for arithmetic as follows: *Given any two positive numbers a and b, of which b is the greater, there exists a certain positive integer n, such that* $a + a + a + \cdots (n \text{ times}) > b$.

Passing over the first few pages of the book, which are intended for very elementary students, we find, on page 35, in direct violation of Archimedes's axiom, the introduction of a certain positive ($\neq 0$) number δ, termed an "infinitely small" number, such that

$$\delta + \delta + \delta + \cdots (n \text{ times}) < 1$$

for any (finite) integer n. And the "fundamental theorem" is repeatedly stated (*e. g.*, page 141) as follows: "The product of a finite quantity and an infinitesimal is always infinitesimal"; while on page 66 it is stated that "the concept of an infinitesimal does not coincide exactly with zero." *

These statements explicitly deny the axiom of Archimedes, though the author does not mention the fact. Nor will the

*See also p. 46; and p. 90, where an infinitesmal unit is considered.

non-euclidean geometer protest against a geometry in which certain usual axioms are expressly violated, provided the fact of such violation is constantly kept in mind and no appeal whatever is made to ordinary geometry. In particular, one axiom which is at the basis of all considerations of continuity in the ordinary sense having been rejected, we naturally expect a thorough research of the axioms of continuity which are assumed to hold. We shall see that the author has unfortunately not adhered to such a programme.

No result of the book can then be accepted, even for a non-euclidean, non-archimedean, geometry. But it is interesting to note that some of the results obtained are such as actually do hold in the non-archimedean geometry actually constructed by Hilbert, in his Grundlagen der Geometrie.

If we consider as our realm of numbers all the rational functions of a certain parameter t, and define "equality," "greater than," and "less than" by the rule $f(t) >, =, < g(t)$ according as $f(t) - g(t) >, =, < 0$ for all sufficiently large positive values of t, then all the axioms of arithmetic (see Hilbert, page 26), except the axiom of Archimedes, are satisfied.*

We find, for instance, in the book in hand, on page 62, that "a ratio of two infinite [or infinitesimal] numbers of the same class is a finite number, * * * and a product of two infinite numbers of the same class is not finite, but is infinite [or infinitesimal] of another class." Arrangement into classes might be made by the rule that numbers of any one class satisfy, among themselves, the axiom of Archimedes. It is then readily seen that the above statement holds true in Hilbert's non-archimedean geometry. The two numbers

$$\frac{a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0}{b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0} \quad \text{and} \quad \frac{c_r t^r + c_{r-1} t^{r-1} + \cdots + c_0}{d_s t^s + d_{s-1} t^{s-1} + \cdots + d_0}$$

are of the same class if $m - n = r - s$. Their quotient is of the class $(m - n) - (r - s) = 0$, and is hence "finite," *i. e.*, in the same class with 1; while their product is of class $(m - n) + (r - s)$ which is neither zero nor equal to $(m - n)$ unless $m - n = 0$. This latter exceptional case corresponds to the statement (page 62): "The class of finite numbers is that in which the product and the ratio of any two numbers of the class again belongs to the class" — which holds for positive

*The form is slightly different from Hilbert's.

numbers. We are also informed that the choice of which class is to be considered finite is not at our disposal.

On page 177 the author says "the motion of sliding"—(presumably continuous motion in euclidean space) "is to be thought of not in general, but only for a certain class of segments at a time." The notion expressed practically amounts to saying that the complete (non-archimedean) geometry assumed does not permit of continuous motion in the ordinary sense. The author is at great pains elsewhere * to explain away the difficulty that a passage from "finite" to "infinite" [or to "infinitesimal"] numbers, for instance, can be affected *neither* by a sudden spring, nor yet *without* such a spring. That he has violated a fundamental axiom of continuity, and hence cannot expect his geometry to be continuous in the ordinary sense, is not made clear.

Again, on page 194, for example, the author is surprised to find himself in difficulties with such words as "continuous," "hole-less," etc., as applied to space. And he declares that a segment (in geometry) or a number (in arithmetic), when added to a (greater) second, "does not alter the value of the second except when it is of the same class." And on the next page (195): "for this reason and only for this reason we shall be correct if we consider in any discussion the numbers (or segments) of any one class." But this is returning to archimedean geometry, and relinquishing our previous violation of the archimedean axiom. And, unfortunately for this assertion, numbers and segments of different classes *are* treated in the same discussion constantly throughout the book (see pages 28, 29, 51, 59, 77, etc.).

Finally we find, on page 196, "the limit of a (finite) variable x is a region (precisely expressed, a region of finiteness); and this region is infinitely small, but otherwise arbitrary."

Such are, indeed, in part, the necessary conclusions of a non-archimedean geometry. But it should be carefully noted that such results have nothing to do with euclidean geometry, *i. e.*, with space as *ordinarily* conceived and treated. It is the author's lack of appreciation of this fact, and his constant appeal to the intuition (and to even much less strict metaphysical persuasion) which robs the book of any value whatever, even as a discussion of non-archimedean geometry. Even the archimedean axiom itself seems to be used implicitly at

* See *e. g.*, p. 10.

points, in assuming that the geometry is continuous in the ordinary sense, but the arguments are so intermingled with intuitional conclusions that it is wholly impossible to decipher just what axioms the author is implicitly assuming. It is scarcely necessary to state that the discussions of limits, infinite series, the parallel axiom,* derivates, etc., are riddled with errors; and the minor matters treated are no less free from gross mistakes. That Leibniz's ideas of infinity are the nearest approach to the ideas of this book, as the author frequently asserts, is in itself a sufficient condemnation.

The moral of the book concerns the thoughtless introduction of "differentials" in our ordinary books on elementary calculus.† A "differential," treated as an "infinitesimal," or "infinitely small quantity," is precisely a non-archimedean number; and no reasoning can be applied to a system of numbers in which such "infinitesimals" occur without an investigation of the extraordinary properties of such a non-archimedean (non-euclidean) arithmetic (or geometry).

On the other hand, if differentials are to be introduced in the calculus, we may do so (in two dimensions) by considering the perfectly finite quantities

$$dx \equiv \Delta x, \quad dy \equiv \frac{dy}{dx} \cdot \Delta x, \quad d_2 y \equiv \frac{d^2 y}{dx^2} \cdot \overline{\Delta x^2},$$

etc., where Δx is a certain fixed finite increment of x chosen at pleasure. Such a treatment can be made perfectly rigorous, and the advantages (if any such exist) of the differential notation can be preserved, while our geometry and arithmetic remain euclidean and archimedean. But the differentials here considered are perfectly finite quantities, with no taint of peculiarity or obscurity.

The errors in conceptions and in modes of thought, induced in a student by the use of differentials in a loose sense, are well illustrated by the nature of the conceptions of this author. And on serious consideration the treatment of calculus by the use of "infinitely small" differentials must be discarded‡ by

* A "proof" of this axiom is given on p. 26. The author states that previous proofs are lacking in rigor!

† And to a minor extent, the ordinary calculus treatment of indeterminate forms, and of infinite series.

‡ This statement of course presupposes that the elementary course on the calculus is to be based on euclidean geometry, and is to use intuitive proofs of theorems involving continuity and limits, to some extent.

any sincere writer as totally illogical in itself, and baneful in its effects upon the student.

While this is but one phase of the book in hand, it is the most important phase, to which all else is made subordinate. We will then note but one other point: namely, that the author protests against a violation of the axiom that two points always determine a straight line (page 44), but assumes that "a whole is greater than a part" even when the "whole" in question is infinite (page 19). In both particulars the modern tendency is certainly quite the opposite.

The philosophical portion of the book will not be criticised here, but it is scarcely felt that a philosopher would care to be sponsor for the statements made — certainly not for the style in which they are presented. The philosophical views of mathematical thought, and in particular of infinity and infinitesimals, must surely take into account, however, the positive results now in the possession of mathematicians regarding the effect of the violation of the archimedean axiom upon our system of axioms and upon our conceptions of space.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
November, 1902.

SOME RECENT GERMAN TEXT-BOOKS IN GEOMETRY.

Lehrbuch der Elementar-Geometrie. Erster Theil, Planimetrie. Von Dr. E. GLINZER. Siebente Auflage. Dresden, Kühtmann, 1899. 4to, 120 pp.

Grundriss der Geometrie. I. *Planimetrie.* Von J. H. KÜHL. Zweite Auflage. Dresden, Kühtmann, 1900. 4to, 95 pp.

Lehrbuch der Stereometrie. Von Dr. P. SAUERBECK. Stuttgart, Kröner, 1900. 4to, 291 pp.

Die Elemente der analytischen Geometrie. Zweiter Theil: Die analytische Geometrie des Raumes. Von Dr. F. RUDIO. Dritte, verbesserte Auflage. Leipzig, Teubner, 1901. 4to, vi + 186 pp.

THE authors of the first two texts are connected with the institution in Hamburg known as the Allegemeine Gewerbeschule und Baugewerkschule. One might expect accordingly

a general disregard of anything approaching euclidean rigor in their treatment of the subject. In fact, no precise statement of axioms or definitions is anywhere to be found in either book. The parallel axiom, as such, does not appear at all. Dr. Glinzer gives Euclid's form, viz.: If two lines are cut by a third so that the sum of the interior angles on one side is less than a straight angle, these lines will, if sufficiently produced, meet on that side — as a *proposition*, the proof following it involving comparison of infinite areas. The trivial nature of this proof is, however, emphasized by the setting in small type. The other author assures the pupil that the proof of the proposition: If two lines are cut by a third under equal alternate interior angles, the lines are parallel, does not necessarily imply the truth of the converse, but the correctness of the latter is easily established graphically.

Irrational numbers receive scant attention, the propositions involving incommensurable lengths being regarded as relatively unimportant. The word "limit" is not mentioned at all, but the pupil is told off-hand to apply the formulæ for regular polygons to the case when the number of sides become infinite.

All of this goes to show that, in this German industrial school, the facts of geometry are to be learned at any cost, the development of the student's logical faculties being a matter of secondary importance. A geometric sense must be cultivated and the feeling instilled in the learner that drawing a figure is often half the battle in proving a theorem in geometry. To this end, much attention is paid to constructive geometry, a feature worthy of imitation in American texts.

In extent the subject matter is much the same as the traditional plane geometry. Type and figures are clear, but hardly up to the standard of our best American books.

In the preface to Dr. Sauerbeck's volume the author registers his opinion that "the first object of solid geometry is the cultivation of space intuition." With this view the reviewer is heartily in agreement, and he believes that the usual American text-book contributes very little to this end. For the stress is felt by the student to be laid on the *form* of proof, and not on the *content* of the proposition. After a strenuous course in plane geometry, it may be assumed that euclidean form and rigor in demonstration have been acquired, and in the remainder of the course, emphasis should be put not on logical develop-

ment and sequence of propositions, but on the facts of solid geometry and spatial relations. The traditional course in American colleges supplementing the usual requirement in plane geometry for this reason results in a paltry addition to the sum of the student's knowledge of geometry. How much greater would be the gain in power if an amount of ground could be covered similar to that presented in Dr. Sauerbeck's text! For in pages 1–59 the author discusses the usual theorems involving metrical relations of planes and lines, but in addition such of the elements of projective geometry as naturally arise in the consideration of central projection, the principle of duality, etc. Next follows a section on crystallography, well done but of doubtful value. Section V, pages 84–133, treating of the polyhedral angle and the sphere, is quite complete, including radical planes, axes, centers of similitude, etc. Following this (pages 133–211) surfaces of revolution are discussed, and in this section the subject of conic sections finds consideration, together with stereographic projection and applications. The closing pages treat of mensuration. In these the theorem of Cavalieri is assigned its proper place beside the method of exhaustion. One wonders why this extremely simple, evident and powerful theorem has won no place in American texts. The theorems of Guldin are proven, but seem out of place, at least in their general form. The very general class of solids known as Simpson's solids furnish an excellent example in cubature, and give, as analogous formulæ in the plane, the well-known Simpson's rule for quadrature. A commendable feature of the book is the large number of exercises, solved and unsolved.

To repeat, the position of the author seems to the reviewer a very strong one, and it is to be hoped that the coming years will show a departure from the present prevalent practice of teaching solid geometry in our colleges after the manner of the secondary schools.

Dr. Rudio's book, with its companion volume on the plane, is deservedly popular in Germany. Evidence of great care and thoroughness in writing is everywhere abundant throughout the volume, and the result is a most excellent text. The principle of projection is adopted most wisely as fundamental, and the fixing of the algebraic sign of lengths, areas and volumes carefully treated. The author has kept strictly within the

limits set by the title, the book closing with a discussion of the elementary properties of quadrics, such as tangent planes, diameters, rectilinear generators, etc. The collection of exercises is large, and figures and type are clear and attractive.

PERCEY F. SMITH.

SHEFFIELD SCIENTIFIC SCHOOL,
December, 1902.

NOTES.

THE opening (January) number of volume 4 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Orthocentric properties of the plane n-line," by FRANK MORLEY; "Definitions of a field by independent postulates," by L. E. DICKSON; "Definitions of a linear associative algebra by independent postulates," by L. E. DICKSON; "Two definitions of a commutative group by sets of independent postulates," by E. V. HUNTINGTON; "Definitions of a field (Körper) by sets of independent postulates," by E. V. HUNTINGTON; "On the invariants of differential forms of degree higher than two," by C. N. HASKINS; Ueber die Reducibilität der Gruppen linearer homogener Substitutionen," by ALFRED LOEWY; "The quartic curve as related to conics," by A. B. COBLE; "The cogredient and digredient theories of multiple binary forms," by EDWARD KASNER; "On the envelopes of the axes of a system of conics passing through three points," by R. E. ALLARDICE; "A Jordan curve of positive area," by W. F. OSGOOD.

THE January number (volume 25, number 1) of the *American Journal of Mathematics* contains: "The parametric representation of the tetrahedroid surface," by D. N. LEHMER; "On ternary monomial substitution groups of finite order with determinant ± 1," by E. B. SKINNER; "On forms of unicursal sextic scrolls," and "On forms of sextic scrolls of genus one," by VIRGIL SNYDER; "Note on symmetric functions," by E. D. ROE, Jr.

THE number contains a portrait of Professor L. CREMONA.

THE lectures delivered by Professor OSKAR BOLZA before the colloquium of the AMERICAN MATHEMATICAL SOCIETY at Ithaca, N. Y., in August, 1901, will be published early in the

spring as one of the supplementary volumes of the Decennial publications of the University of Chicago, under the title: "Lectures on the calculus of variations."

THE committee on definitions of college entrance requirements in mathematics, appointed by the AMERICAN MATHEMATICAL SOCIETY at the summer meeting held at Evanston, September, 1902, presented to the Council, on December 29, a tentative report, in the form of a memorandum. In this memorandum are defined the requirements in algebra, plane geometry, solid geometry, higher algebra, and plane trigonometry. Algebra covers the elementary subject through the progressions, with the note that a college finding it impracticable to meet this normal requirement is advised to demand algebra through quadratics, and that a college preferring to hold two examinations is advised to make the division immediately before quadratic equations. Plane geometry is defined as including the usual theorems and constructions of good standard text-books, the solution of original exercises, applications to problems of mensuration of lines and of plane figures, and to loci problems. Solid geometry is similarly defined. Higher algebra is taken to cover permutations and combinations, applications of the principle of mathematical induction, theory and use of logarithms (not involving infinite series), determinants, elements of the theory of equations (with graphic methods), Horner's method, elements of the theory of complex numbers (with graphic representation, limited to sums and differences). Plane trigonometry is defined to include the six functions as ratios; the proofs of principal formulas, in particular the sine, cosine and tangent of $A \pm B$ and of $2A$, and the product expressions for the sum or the difference of two sines or of two cosines; the properties of logarithms and the use of tables; the solution of triangles, with applications.

THE first winter meeting of the American association for the advancement of science was held at Washington, D. C., during convocation week, December 29, 1902, to January 3, 1903. Among the officers elected for the present year are: President of the Association, CARROLL D. WRIGHT; Vice-President of Section A, A. H. TITTMANN; Secretary of Section A, L. G. WELD. The next meeting of the Association will be held at St. Louis in convocation week, 1903–1904.

At the meeting of the London mathematical society held on December 11, 1902, the following papers were read: By Professor L. E. Dickson: (1) "The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field;" (2) "Generational relations of an abstract simple group of order 4080;" by Dr. H. F. Baker: (1) "On the calculation of the finite equations of a continuous group," (2) "On the integration of linear differential equations," (3) "On some cases of matrices with linear invariant factors;" by Professor M. J. M. Hill: "The continuation of the power series for arcsin x;" by Mr. E. T. Whittaker: "The functions associated with the parabolic cylinder in harmonic analysis;" by Mr. H. M. Macdonald: "Some applications of Fourier's theorem;" by Rev. F. H. Jackson: "Series connected with the enumeration of partitions;" by Mr. W. H. Young: "Sets of intervals, part ii: overlapping intervals;" By Mr. G. H. Hardy: "On the expression of the double zeta and gamma functions in terms of elliptic functions;" by Mr. J. H. Grace: "Perpetuants (second paper)."

Oxford University.—The following courses in mathematics are announced for Hilary term, 1903: By Professor W. Esson: Comparison of analytic and synthetic methods in the geometry of conics, two hours; Synthetic geometry of cubics, one hour.—By Professor E. B. Elliott: Elements of elliptic functions, two hours.—By Professor A. E. H. Love: Attractions and electrostatics, two hours; Theory of potential, one hour.—By Mr. A. L. Dixon: Calculus of finite differences, one hour.—By Mr. J. E. Campbell: Algebra of quantics, one hour.—By Mr. P. J. Kirkby: Higher plane curves, two hours.—By Mr. C. H. Sampson: Solid geometry (continued), two hours.—By Mr. C. E. Haselfoot: Theory of equations, one hour.—By Mr. J. W. Russell: Pure geometry, two hours.—By Mr. C. Leudesdorf: Geometry (maxima and minima, inversion, etc.), two hours.

At the annual public meeting of the Paris academy of sciences, held on December 22, 1902, the following mathematical prizes were awarded: Grand prize, 3,000 francs, Ernest Vessiot; very honorable mention, Jean Le Roux; Bordin prize, 3,000 francs, not awarded; very honorable mention, W. de Tannenberg; Francœur prize, 1,000 francs, Émile Lemoine; Poncelet prize, 2,000 francs, Maurice d'Ocagne;

Extraordinary prize of 6,000 francs, divided between M. ROMAZOTTI and M. DRIENCOURT.

THE International congress of historical sciences, which was projected to be held at Rome in April, 1902, but was afterward postponed, will meet in that city on April 2–9, 1903. Eight sections have been arranged, of which the last is devoted to the history of mathematical, physical, natural and medical sciences. The invitations to participate in section VIII are issued by a committee of which Professor Gino Loria of Genoa is the representative of the interests of pure mathematics. Information concerning the congress may be obtained by addressing the Segretariato generale del congresso internazionale di scienze storiche, Collegio Romano, via del Collegio Romano, 26, Rome.

THE press of Girardi and Audebert, of Dole, France, has recently issued a circular of information, in English, for the benefit of American and other foreign students who contemplate attending the French universities. The circular covers particularly the questions of matriculation and degrees, but it also gives valuable information concerning the various university centers. The Report, just issued, of the U. S. Commissioner of Education, 1901, volume 1, page 1113, gives similar information.

THE first annual *Sitzungsbericht* of the Berlin mathematical society, has been reprinted from the *Archiv der Mathematik und Physik*, and published separately. Presumably it is intended to issue these proceedings annually in this form.

THE following academic appointments and promotions have recently been announced: DR. ALBERT BENTELI has been appointed professor of geometry in the University of Bern. Dr. A. SUCHARDA has been made professor of mathematics in the Bohemian Technical high school at Brünn. Dr. V. VARICAK has been made professor of mathematics in the University of Agram. M. GUITTON, of the lycée at Caen, has been appointed to a lectureship in mathematics in the faculty of sciences in that city; M. ZORETTI to an instructorship in mathematics in the Ecole normale supérieure, Paris.

AMONG recently elected foreign members of the Royal Society are Dr. G. W. HILL and Professor GASTON DARBOUX. The Society has awarded its medal to Professor HORACE LAMB for his investigations in mathematical physics.

PROFESSOR LUIGI CREMONA, of Rome, has been elected a foreign member of the American academy of arts and sciences, of Boston.

PROFESSOR C. SEGRE, of Turin, has been elected an honorary fellow of the Cambridge philosophical society.

MR. G. T. WALKER, fellow and lecturer of Trinity College, Cambridge, has been appointed director-general of the Indian meteorological bureau.

LIEUTENANT MAX GENTY, known through numerous mathematical contributions, died at Toulon on October 24, aged 35 years.

THE name of the eminent Russian mathematician whose death was noticed on page 223 of the January number of the BULLETIN should have been printed Professor NIKOLAUS BUGAIEV.

RECENT catalogue of second-hand mathematical works: Fr. Strobel, Jena; general science, including mathematics, 1012 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABEL (N. H.). Festskrift ved Hundredaarsjubilaeet for Niels Henrik Abels Födsel. Christiania, 1902. 4to. 357 pp. 2 portraits, 6 facsimiles. M. 14.50

ABHANDLUNGEN zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor. Heft 14. Inhalt: A. A. Björno, Studien über Menelaos' Sphärik; Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen; H. Suter, Nachträge und Berichtigungen zu "Die Mathematiker und Astronomen der Araber und ihre Werke"; K. Bopp, Antoine Arnauld, der grosse Arnauld, als Mathematiker. Leipzig, Teubner, 1902. 8vo. 8 + 338 pp.

AHL (F.). Untersuchungen über geodätische Linien. Kiel, 1901. 8vo. 50 pp. M. 1.50

BJÖRNO (A. A.). See ABHANDLUNGEN.

BOPP (K.). See ABHANDLUNGEN.

GEIGENMÜLLER (R.). Leitfaden und Aufgabensammlung zur höheren Mathematik; für technische Lehranstalten und den Selbstunterricht bearbeitet. Vol. I: Die analytische Geometrie der Ebene und die algebraische Analysis. 6te Auflage. Mittweida, Polytechnische Buchhandlung, 1902. 8vo. 7 + 302 pp. Cloth. M. 6.50

GERNET (N.). Untersuchung zur Variationsrechnung. Ueber eine neue Methode in der Variationsrechnung. (Diss.) Göttingen, 1902. 8vo. 75 pp.

GREINER (A.). Ueber orthogonale Invarianten der Kurven dritter Ordnung mit unendlich fernem Doppelpunkt und ihre geometrische Bedeutung. (Diss.) Jena, 1902. 8vo. 41 pp.

HEDRICK (E. R.). The English and French translations of Hilbert's Grundlagen der Geometrie. 8vo. 1902. (*Bulletin of the American Mathematical Society* (2) 9, pp. 158–165.)

HETTNER (G.). See WEIERSTRASS (K.).

KNOBLAUCH (J.). See WEIERSTRASS (K.).

KOENIGSBERGER (L.). Hermann von Helmholtz. Vol. I. Braunschweig, Vieweg, 1902. 8vo. 12 + 375 pp., 3 portraits. M. 8.00

KUTTA (W.). Beitrage zur näherungsweisen Integration totaler Differentialgleichungen. (Diss.) München, 1901. 8vo. 19 pp.

LOREY (W.). Ueber das geometrische Mittel, insbesondere über eine dadurch bewirkte Annäherung kubischer Irrationalitäten. (Diss.) Halle, 1902. 4to. 27 pp.

LUDWIG (W.). Die Horopterkurve, mit einer Einleitung in die Theorie der kubischen Raumkurve. Halle, Schilling, 1902. 8vo. 36 pp. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling in Halle a. S. Neue Folge, No. 3.) M. 1.00

NERNST (W.) und SCHÖNFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 3te Auflage. München und Berlin, Oldenbourg, 1902. 8vo. M. 10.00

PASCAL (E.). I gruppi continui di trasformazioni; parte generale della teoria. Milano, Hoepli, 1903. 16mo. 11 + 358 pp. (Manuali Hoepli.)

——. Lezioni di calcolo infinitesimale. Parte II: Calcolo integrale. 2a edizione, completamente riveduta. Milano, Hoepli. 1902. 16mo. 8 + 329 pp. (Manuali Hoepli.)

PERRY (N.). Das Problem der konformen Abbildung für eine specielle Kurve von der Ordnung $3n$. (Diss.) München, 1901. 8vo. 34 pp.

ROUGUET (V.). Etude géométrique des surfaces dont les lignes de courbure d'un système sont planes et égales. Marseille, 1902. 4to. 43 pp.

SCHÖNFLIES (A.). See NERNST (W.).

STEPHANSEN (M. A. E.). Ueber partielle Differentialgleichungen 4ter Ordnung, die ein intermediäres Integral besitzen. Kristiania, Cammermeyer, 1902. 8vo. 80 pp. M. 2.80

SUTER (H.). See ABHANDLUNGEN.

WEIERSTRASS (K.). Mathematische Werke. Vol. IV: Vorlesungen über die Theorie der Abelschen Transcendenten. Bearbeitet von G. Hettner und J. Knoblauch. Berlin, Mayer & Müller, 1902. 4to. 14 + 632 pp. M. 40.00

WÖRNER (K.). Ueber eine besondere Gattung von Gruppen. Freiberg, 1902. 8vo. 36 pp. M. 1.20

II. ELEMENTARY MATHEMATICS.

AGOLINI UGOLINI (G.). La geometrica od esatta divisione sessagesimale della circonferenza, ossia la scoperta dell'angolo di un grado; geometrica inscrizione nel circolo dei poligoni regolari di 9 e 45 lati e loro consequenti; dimostrazione matematica. Sanseverino-Marche, Bellabarba, 1902. 8vo. 15 pp.

ASHTON (C. H.) and MARSH (W. R.). Plane and spherical trigonometry; an elementary text-book. New York, Scribner, 1902. 8vo. 10 + 157 pp. Cloth.

BARDEY (E.). Anleitung zur Auflösung eingekleideter algebraischer Aufgaben. 2te, völlig umgearbeitete Auflage von F. Pietzker. Leipzig, Teubner, 1903. 8vo. 8 + 159 pp. Cloth. M. 2.60

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